Motion of a random walker in a quenched power law correlated velocity field

Soumen Roy[1,*] and Dibyendu Das[2,†]
[1]*Institute of Physics, Bhubaneswar 751005, India*
[2]*Department of Physics, Indian Institute of Technology Bombay, Powai, Mumbai 400076, India*
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We study the motion of a random walker in one longitudinal and d transverse dimensions with a quenched power law correlated velocity field in the longitudinal x direction. The model is a modification of the Matheron–de Marsily model, with long-range velocity correlation. For a velocity correlation function, dependent on transverse coordinates $\vec{y}$ as $1/(a+|\vec{y}_1-\vec{y}_2|)^{\alpha}$, we analytically calculate the two-time correlation function of the x coordinate. We find that the motion of the x coordinate is a fractional Brownian motion (FBM), with a Hurst exponent $H=\max[1/2,(1-\alpha/4),(1-d/4)]$. From this and known properties of FBM, we calculate the disorder averaged persistence probability of $x(t)$ up to time t. We also find the lines in the parameter space of d and α along which there is marginal behavior. We present results of simulations which support our analytical calculation.

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I. INTRODUCTION

It is well known that a random walker making short-range hops, namely, a Brownian particle in a thermal environment, has an average incremental displacement which varies as the square root of the time difference. Although this property is rather robust, deviation from this diffusive behavior has been observed in a wide variety of physical situations [1]. Superdiffusion is well known in the context of turbulence and Lévy walks [2]. Yet another way to see anomalous diffusion is to subject the random walker to a quenched velocity field. A concrete example of this is the Matheron–de Marsily (MdM) model, which was introduced to study the hydrodynamic dispersion of solute particles in sedimentary layered porous rock formations [3]. In the MdM model a single particle diffuses in a $(1+1)$-dimensional layered medium, such that the disorder is anisotropic. The motion of the particle is purely Brownian along the transverse y direction, while along the longitudinal x direction, added to the thermal noise, the particle is driven by a quenched drift velocity $v(y)$ that is a random function of only the transverse coordinate y. The coupling between diffusion and convection due to the spatially random but temporally static velocity field generates a typical bias in the x direction giving rise to a superdiffusive longitudinal transport for large t [1,4–6].

The temporal behavior is quantitatively described by the standard two-time dependent correlation function, and the persistence probability. Intuitively, persistence of a system concerns the property of the system to stay in some given state. The persistence probability $P(t)$ which is related to the first-passage probability $F(t)$, as $F(t)=-dP(t)/dt$ is simply the probability that the particle does not cross a given point up to time t. In systems with superdiffusion, $P(t)$ is expected to get enhanced. Persistence properties in disordered systems are harder to find, and thus results known for such models as the Sinai model [7–9] and the MdM model [6,10] are of great importance. Recently for a $(d+1)$-dimensional version of the MdM model the persistence probability was analytically derived in [11].

In earlier studies of the MdM model [4,6,10,11], only short-range correlated $v(y)$ was considered. However, since originally the model was motivated [3] by general flow in certain types of fractured rocks with parallel fractures which allow the propagation of dissolved species, it is possible that the flows parallel to the bedding might have velocity correlations that are spread over long distances. It may be mentioned that different aspects of dynamics in MdM fields with algebraic correlations have been studied [12]. Generally long-range correlated disordered problems are hard to solve. For isotropic disorder with power law correlations, various results are known [1,13,14]. The MdM model being anisotropic allows for exact analytic treatment as compared to the field theoretic treatment done in other cases. In this paper we have been able to analytically derive the incremental correlation function and persistence probability, for the MdM model with long-range *power law* correlated $v(y)$. We find that long-range correlation incorporates additional temporal behavior, as compared to the short-range MdM model. The relevant exponent of $P(t)$ becomes dependent on the power law exponent α and the transverse dimension d.

The method that we follow is similar to [11,15] and was used even earlier in [16]. We first show that $x(t)$ is a fractional Brownian motion (FBM), and then use its known first-passage property to solve for the persistence probability. A stochastic process $x(t)$ [with zero mean $\langle x(t)\rangle=0$] is called a FBM if its incremental two-time correlation function $C(t_1,t_2)=\langle[x(t_1)-x(t_2)]^2\rangle$ is stationary, i.e., depends only on the difference $|t_1-t_2|$, and, moreover, grows asymptotically as a power law,

$$C(t_1,t_2) \sim |t_1-t_2|^{2H}, \quad |t_1-t_2| \gg 1. \tag{1}$$

The parameter H is called the Hurst exponent which characterizes the FBM [17] and $\langle\cdots\rangle$ denotes the expectation value

*Electronic address: sroy@iopb.res.in
†Electronic address: dibyendu@phy.iitb.ac.in

over all realizations of the process. For example, the ordinary Brownian motion is a FBM with $H=1/2$. The zero-crossing properties of FBM have been studied before [16,18–20] and it is known, both analytically and numerically, that the probability that a FBM does not cross zero up to time t has a power law decay $P(t)\sim t^{-\theta}$ for large t with $\theta=1-H$ [16,20]. We use this property to derive θ from H.

II. MODEL AND MAIN RESULTS

We consider a generalized version of the MdM model in d (>0) transverse plus one longitudinal spatial dimensions. The transverse coordinates of the particle, denoted by $y_i(t)$ with $i=1,2,\ldots,d$, undergo ordinary Brownian motion,

$$\dot{y}_i = \eta_i(t), \tag{2}$$

where the η_i's are zero-mean Gaussian white noises with correlators $\langle \eta_i(t)\eta_j(t')\rangle=\delta_{i,j}\delta(t-t')$. Equivalently we can use a single equation involving d-dimensional vectors $\vec{y}=\{y_i(t)\}$ and $\vec{\eta}=\{\eta_i(t)\}$:

$$\dot{\vec{y}} = \vec{\eta}(t). \tag{3}$$

The longitudinal coordinate $x(t)$, in contrast, is driven by a random drift $v[\vec{y}(t)]$ that depends only on the transverse coordinates $\vec{y}(t)$,

$$\dot{x} = v[\vec{y}(t)] + \eta_x(t), \tag{4}$$

where η_x is again a zero-mean Gaussian white noise with $\langle \eta_x(t_1)\eta_x(t_2)\rangle=\delta(t_1-t_2)$. The velocity field $v(\vec{y})$ is quenched, i.e., constant in time. Operationally this means that for a given realization of the function v, one has to evolve $x(t)$ over "thermal histories" in Eq. (4) and then one needs to "disorder average" (denoted by an $\overline{\cdots}$) over different realizations of the random function v. In the original MdM model and in the later work on slightly modified versions of it [1,3–6,10,11], the random function $v(\vec{y})$ has been considered to be a Gaussian with zero mean ($\overline{v(\vec{y})}=0$), with a correlator $\overline{v(\vec{y}_1)v(\vec{y}_2)}$, which is either $\delta(\vec{y}_1-\vec{y}_2)$ or short ranged, namely, $1/[2\pi da^2]^{d/2}\exp[-(\vec{y}_1-\vec{y}_2)^2/2da^2]$ (where a is a small cutoff).

In contrast to the above, here we consider the random function v to have long-range power law correlation:

$$\overline{v(\vec{y})} = 0,$$

$$\overline{v(\vec{y}_1)v(\vec{y}_2)} = \frac{1}{(a+|\vec{y}_1-\vec{y}_2|)^{\alpha}}. \tag{5}$$

The exponent $\alpha>0$ denotes the strength of the correlation function, and the short-distance cutoff scale a ensures that there is no divergence as $|\vec{y}_1-\vec{y}_2|\to 0$. This model is analytically treated in Sec. III and numerically in Sec. IV. We will see that the role played by the scale a is very interesting. For $\alpha<d$, it is irrelevant and can be set to zero, but for $\alpha>d$, it has to be finite to ensure a well-behaved asymptotic behavior.

Our main result is that the thermal and disorder averaged two-time correlation function of the x coordinate, $\overline{\langle[x(t_1)-x(t_2)]^2\rangle}\approx|t_2-t_1|^{2H}$. Thus the process $x(t)$ is a fBM. The Hurst exponent $H=\max[1/2,(1-\alpha/4),(1-d/4)]$. This means that for $d<2$, the exponent will change from $H=(1-\alpha/4)$ for $\alpha<d$, to $H=(1-d/4)$ for $\alpha>d$. On the other hand, for $d\geq 2$, $H=(1-\alpha/4)$ for $\alpha<2$, and $H=1/2$ for $\alpha>2$. It follows from this that the persistence probability of $P(t)\sim t^{-\theta}$, with $\theta=1-H=\min[1/2,\alpha/4,d/4]$. We have also found the lines in d and α plane along which there is marginal behavior.

III. TWO-TIME CORRELATION FUNCTION AND PERSISTENCE PROBABILITY

The analytical treatment that we present here is similar to that of [11,15] up to the point that the explicit power law form of velocity correlation is substituted. Nevertheless, for completeness, we give the steps briefly. In order to solve for correlation functions of $x(t)$ from Eq. (4), we need knowledge of correlation functions of $\vec{y}(t)$. By integrating Eq. (3) we get

$$\vec{y} = \int_0^{\tau} \vec{\eta}(\tau)d\tau. \tag{6}$$

From the above linear relation we conclude that since $\vec{\eta}$ is a Gaussian random variable, so is $\vec{y}$. Moreover, another vector $\vec{Y}=\vec{y}(\tau_1)-\vec{y}(\tau_2)$ which will be useful in the following calculations is also a Gaussian random variable by virtue of the linear relation. Using Eq. (6),

$$\vec{Y} = \int_0^{\tau_1} \vec{\eta}(\tau)d\tau - \int_0^{\tau_2} \vec{\eta}(\tau)d\tau, \tag{7}$$

and it follows that $\langle\vec{Y}\rangle=0$ and variance $\langle\vec{Y}^2\rangle=d|\tau_1-\tau_2|$, since $\langle \eta_i(t)\eta_j(t')\rangle=\delta_{i,j}\delta(t-t')$. Thus explicitly the distribution

$$P(\vec{Y}) = (2\pi d|\tau_1-\tau_2|)^{-d/2}\exp(-\vec{Y}^2/2d|\tau_1-\tau_2|). \tag{8}$$

With the above solution of the transverse coordinates $\vec{y}(t)$, we now proceed to treat the longitudinal coordinate $x(t)$. We note that the averages so far over the transverse coordinates, namely $\langle\ldots\rangle$, were thermal averages. The coordinate $x(t)$ depends on both a thermal noise η_x and a quenched random noise v. So we need to do a thermal as well as a quenched disorder average, which we will denote by $\overline{\langle\ldots\rangle}$, where in particular the $\overline{\cdots}$ denotes the quenched disorder average. Integrating Eq. (4) we get

$$x(t) = \int_0^t \eta_x(\tau)d\tau + \int_0^t v[\vec{y}(\tau)]d\tau. \tag{9}$$

From Eq. (9) we get $\overline{\langle x(t)\rangle}=0$, while using $\langle \eta_x(t_1)\eta_x(t_2)\rangle=\delta(t_1-t_2)$, we get

$$\overline{\langle x(t_1)x(t_2)\rangle} = \min(t_1,t_2) + I(t_1,t_2), \tag{10}$$

where the integral $I(t_1,t_2)$ is given by

$$I(t_1,t_2)=\int_0^{t_1}\int_0^{t_2}\overline{\langle v[\vec{y}(\tau_2)]v[\vec{y}(\tau_2)]\rangle}d\tau_1 d\tau_2$$
$$=\int_0^{t_1}\int_0^{t_2}\left\langle\frac{1}{(a+|\vec{Y}|)^\alpha}\right\rangle d\tau_1 d\tau_2. \tag{11}$$

In the above, we have used the correlator of v from Eq. (5) and $\vec{Y}$ is the same as in Eq. (7). Since we know the probability distribution $P(\vec{Y})$ explicitly from Eq. (8), we can proceed from Eq. (11) to evaluate

$$I(t_1,t_2)=\int_0^{t_1}\int_0^{t_2}\int_{-\infty}^{\infty}d\tau_1 d\tau_2\frac{1}{(|\vec{Y}|+a)^\alpha}\frac{e^{-|\vec{Y}|^2/2d|\tau_1-\tau_2|}}{(2\pi d|\tau_1-\tau_2|)^{d/2}}d^d\vec{Y}$$
$$=\frac{2}{\Gamma(d/2)}\int_0^{t_1}\int_0^{t_2}\int_0^{\infty}\frac{d\tau_1 d\tau_2 e^{-Z^2}Z^{d-1}dZ}{(Z\sqrt{2d|\tau_1-\tau_2|}+a)^\alpha}$$
$$=\frac{2}{\Gamma(d/2)}\int_0^{\infty}e^{-Z^2}Z^{d-1}J(Z,t_1,t_2)dZ. \tag{12}$$

In Eq. (12) we have substituted $|\vec{Y}|=Z\sqrt{2d|\tau_1-\tau_2|}$ in going from the first line to the second, and then performed the two time integrals exactly to get explicitly the function

$$J(Z,t_1,t_2)=\int_0^{t_1}\int_0^{t_2}\frac{d\tau_1 d\tau_2}{(Z\sqrt{2d|\tau_1-\tau_2|}+a)^\alpha} \tag{13}$$

$$=\frac{1}{Z^2 d}\frac{2a^{2-\alpha}\min(t_1,t_2)}{(2-\alpha)(1-\alpha)}+\frac{(a+Z\sqrt{2t_1d})^{4-\alpha}+(a+Z\sqrt{2t_2d})^{4-\alpha}-a^{4-\alpha}-(a+Z\sqrt{2d|t_2-t_1|})^{4-\alpha}}{Z^4d^2(2-\alpha)(4-\alpha)}$$
$$+\frac{(3-2\alpha)[-a(a+Z\sqrt{2t_1d})^{3-\alpha}-a(a+Z\sqrt{2t_2d})^{3-\alpha}+a^{4-\alpha}+a(a+Z\sqrt{2d|t_2-t_1|})^{3-\alpha}]}{Z^4d^2(3-\alpha)(1-\alpha)(2-\alpha)}$$
$$+\frac{a^2(a+Z\sqrt{2t_1d})^{2-\alpha}+a^2(a+Z\sqrt{2t_2d})^{2-\alpha}-a^{4-\alpha}-a^2(a+Z\sqrt{2d|t_2-t_1|})^{2-\alpha}}{Z^4d^2(1-\alpha)(2-\alpha)}. \tag{14}$$

Although Eq. (14) has many terms, the two-time correlation function $C(t_1,t_2)=\overline{\langle x^2(t_1)\rangle}+\overline{\langle x^2(t_2)\rangle}-2\overline{\langle x(t_1)x(t_2)\rangle}$ has a dependence only on $t=|t_1-t_2|$, and has the following comparatively simpler expression:

$$C(t)=t+\frac{4}{\Gamma(d/2)}\int_0^{\infty}dZ\,e^{-Z^2}Z^{d-1}\Bigg(\frac{At}{Z^2}+\frac{1}{Z^4}[2B(a+Z\sqrt{2dt})^{4-\alpha}$$
$$-2Ca(a+Z\sqrt{2dt})^{3-\alpha}+2Da^2(a+Z\sqrt{2dt})^{2-\alpha}$$
$$-(2B-2C+2D)a^{4-\alpha}]\Bigg). \tag{15}$$

In Eq. (15) we have used the symbols $A=2a^{2-\alpha}/[d(2-\alpha)\times(1-\alpha)]$, $B=1/[d^2(2-\alpha)(4-\alpha)]$, $C=(3-2\alpha)/[d^2(1-\alpha)(2-\alpha)(3-\alpha)]$, and $D=1/[d^2(1-\alpha)(2-\alpha)]$. Further, Eq. (15) can be written as $C(t)=t+\bar{C}$, where $\bar{C}$ is the integral that has to be analyzed to get the leading temporal behavior.

Defining a new variable of integration $y'=Z\sqrt{2dt}$ and substituting in $\bar{C}$, we get

$$\bar{C}=\frac{4t^{2-d/2}}{\Gamma(d/2)(2d)^{d/2}}\int_0^{\infty}dy'\,e^{-y'^2/2dt}y'^{d-1}\Bigg(\frac{A2d}{y'^2}+\frac{4d^2}{y'^4}$$
$$\times[2B(a+y')^{4-\alpha}-2Ca(a+y')^{3-\alpha}+2Da^2(a+y')^{2-\alpha}$$
$$-(2B-2C+2D)a^{4-\alpha}]\Bigg) \tag{16}$$

It turns out that one has to be very careful to extract the asymptotic behavior of the above integral, because the correlations at scales $y'<a$ and $y'>a$ contribute different power laws, and they compete with each other. It is best to split the integral $\int_0^\infty$ in Eq. (16) into two parts $\int_0^a$ and $\int_a^\infty$. Accordingly $\bar{C}=C_1+C_2$. In C_1 we expand the terms of the integrand in powers of y'/a (with $y'<a$), while in C_2 we expand the terms in powers of a/y' (with $y'>a$).

The C_1 integral with its integrand expanded in a series yields

$$C_1=\frac{4t^{2-d/2}}{\Gamma(d/2)(2d)^{d/2}}\int_0^{a}dy'\,e^{-y'^2/2dt}y'^{d-1}a^{-\alpha}$$
$$\times\left[1-\frac{4\alpha}{15}\frac{y'}{a}+O\left(\frac{y'^2}{a^2}\right)\right]\approx b_1a^{d-\alpha}t^{2-d/2} \tag{17}$$

where $b_1=[4/\Gamma(d/2)(2d)^{d/2}][1/d-4\alpha/15(d+1)+\cdots]$. In obtaining Eq. (17) we have used the fact that for small y' and very large t, we may approximate $e^{-y'^2/2dt}\approx 1$ inside the integral. In Eq. (17) the factor $a^{d-\alpha}$ smoothly goes to zero if $a\to 0$ for $\alpha<d$. If we wish, by taking smaller and smaller cutoffs a, we can actually ignore the contribution of C_1 with respect to C_2. On the other hand, if $\alpha>d$, $a\to 0$ will lead to a divergence, and hence we have to keep a finite, and thus the contribution of C_1 cannot be ignored.

Now we turn to the integral C_2 and it goes as follows:

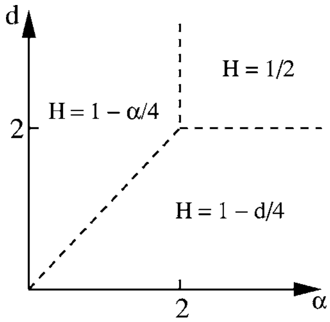

FIG. 1. The values of the exponent H in the three regions of the parameter space of d versus α are shown.

$$C_2 = \frac{4t^{2-d/2}}{\Gamma(d/2)(2d)^{d/2}} \int_a^\infty dy' e^{-y'^2/2dt} y'^{d-1} \Bigg\{ \frac{A2d}{y'^2} + \frac{4d^2}{y'^\alpha} \times \Bigg[2B + 2B(4-\alpha) - 2C\frac{a}{y'} + O\left(\frac{a^2}{y'^2}\right) - (2B - 2C + 2D)\frac{a^{4-\alpha}}{y'^{4-\alpha}} \Bigg] \Bigg\} \approx a_0 t + a_1 t^{2-\alpha/2} + O(t^{3/2-\alpha/2}) + b_2 a^{d-\alpha} t^{2-d/2} \quad (18)$$

where the newly defined constants are $a_0 = 4A/[\Gamma(d/2)(d-2)]$, $a_1 = 32Bd^2/[\Gamma(d/2)(d-\alpha)(2d)^{\alpha/2}]$, $b_2 = 4/\Gamma(d/2) \times \{-4/[(d-2)(1-\alpha)(2-\alpha)] - 8Bd^2/(d-\alpha) + O(a) + b_4\}$, and $b_4 = 24/[(d-4)(1-\alpha)(2-\alpha)(3-\alpha)(4-\alpha)]$. In going from the first to the second relation in Eq. (18) we have made the approximation of replacing the upper limit ∞ by $\sqrt{2dt}$, as the exponential factor $\exp(-y'^2/2dt)$ sharply cuts off the integral beyond that scale anyway. After that we have set $e^{-y'^2/2dt} \approx 1$ and then done the integrals. We see that in Eq. (18), C_2 has three dominant power laws with powers equal to 1, $(1-\alpha/2)$, and $(1-d/2)$. Combining Eqs. (17) and (18), we get finally the two-time incremental correlation function from Eq.(15) as

$$C(t_1,t_2) = C(t) \approx B_1 t + a_1 t^{2-\alpha/2} + B_2 t^{2-d/2}, \quad (19)$$

where $B_1 = 1 + a_0$, and $B_2 = (b_1 + b_2)a^{d-\alpha}$. This implies that asymptotically, at long time t, $C(t) \sim t^{2H}$. Thus $x(t)$ is a FBM process and the associated Hurst exponent H is given by

$$H = \max[1/2, (1-\alpha/4), (1-d/4)]. \quad (20)$$

More explicitly, it means that for $d<2$, for $\alpha<d$, $H=(1-\alpha/4)$ while for $\alpha>d$, $H=(1-d/4)$. On the other hand, for $d\geq 2$, for $\alpha<2$, $H=(1-\alpha/4)$ while for $\alpha>2$, $H=1/2$. This scenario is shown in Fig. 1.

We have analyzed the cases of marginal behavior separately. For the short-range disordered MdM model, $d=2$ was shown [11] to have marginal behavior. For the current model, we have done explicit calculations (similar to those above) for the cases $\alpha=2$, $d=2$, and $\alpha=d$ [21]. (i) For $\alpha=2$, for large t, $C(t) \sim t \ln t$ (for $d>2$), $\sim t^{2-d/2}$ (for $d<2$), and $\sim t(\ln t)^2$ (for $d=2$). This means that the semi-infinite line $\alpha=2$ with $d \geq 2$ (see Fig. 1) has marginal behavior. (ii) For $d=2$, $C(t) \sim t \ln t$ (for $\alpha>2$) and $\sim t^{2-\alpha/2}$ (for $\alpha<2$). Hence the semi-infinite line $d=2$ with $\alpha>2$ (Fig. 1) exhibits marginality. (iii) Finally we also find a logarithmic correction, namely, $C(t) \sim t^{2-d/2} \ln t$, along the line $\alpha=d$, for α and d both <2.

From the known first-passage property of FBM mentioned earlier, the disorder and thermal averaged probability that the process $x(t)$ does not cross zero up to time t decays as a power law $P(t) \sim t^{-\theta}$ with $\theta = 1-H$. Using the results for H in Eq. (20), we get

$$\theta = \min(1/2, \alpha/4, d/4). \quad (21)$$

This means that for $d<2$, if $\alpha<d$, $\theta=\alpha/4$ while if $\alpha>d$, $\theta=d/4$. For $d\geq 2$, if $\alpha<2$, $\theta=\alpha/4$, while if $\alpha>2$, $\theta=1/2$.

Although for $d<2$ and $\alpha>d$, we expect analytically $C(t) \sim t^{2-d/2}$ for large t, we find numerically (see Sec. IV) that it is rather tricky to find this power law. The latter is a consequence of Eq. (17), which tells us that this long-time behavior arises, predominantly, as a contribution of correlations at distance scales $y'<a$. On the other hand Eq. (18) shows that the power law $t^{2-\alpha/2}$ arises out of a contribution due to correlations at scale $y'>a$. So if a finite cutoff a is not kept, and the distance scales $y' \ll a$ are not probed carefully, numerically one may find a wrong asymptotic power, namely, $2H=(2-\alpha/2)$ instead of the expected power $2H=(2-d/2)$. We will discuss this further in Sec. IV.

IV. NUMERICAL SIMULATION AND RESULTS

Since the exponents in Eq. (20) followed from Eq. (19) which is an approximate asymptotic form of the exact Eq. (15), we first checked by numerically integrating Eq. (15) that the predicted powers are indeed true. While doing this, we realized that in order to see the power $C(t) \sim t^{2-d/2}$ in the regime $\alpha>d$ (for $d<2$) one should be very careful to choose the numerically finite integration measure $\delta Z \ll 1$ (where δZ approximates infinitesimal dZ), in order to see the asymptotics. This is related to the comment made earlier that we have to probe $y' \ll a$ (where $y' = Z\sqrt{2dt}$ as mentioned earlier) to get the asymptotic contribution, in this regime.

We now turn to verifying the analytical predictions of the previous section, namely, H as in Eq. (20) and the corresponding exponent θ as in Eq. (21), by doing numerical simulations. We use the time discretised version of Eqs. (4) and (2),

$$y_i(t_{m+1}) = y_i(t_m) + \sqrt{\Delta t}\,\eta_i(t_m), \quad (22)$$

$$x(t_{m+1}) = x(t_m) + \Delta t v(x(t_m)) + \sqrt{\Delta t}\,\eta_x(t_m), \quad (23)$$

where $t_m = m\Delta t$, with m being an integer. We choose $\Delta t < 0.5$ in our simulations so that the stability is guaranteed [16]. The variables $\eta_i(t_m)$ and $\eta_x(m)$ are independent Gaussian variables for all t_m and each is distributed with zero mean and unit variance. Moreover our simulations here are all for $d=1$, i.e., we have a single transverse coordinate y. Hence the subscript i assumes a single value equal to 1.

Unlike [11,15], here we had to numerically create a quenched power law correlated velocity $v(y)$ of the form given by Eq. (5). We note that if we have a δ-correlated

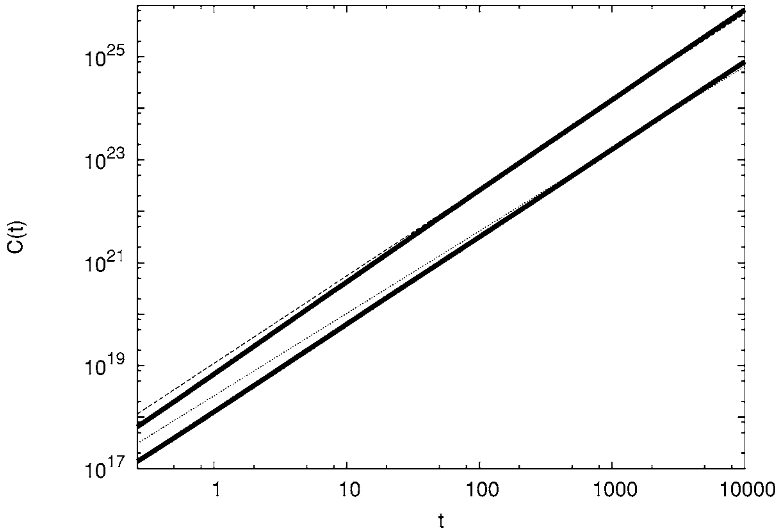

FIG. 2. Log-log plot of $C(t)$ versus t for values of α=0.6 (above) and 0.8 (below). The thin dotted straight lines are fits to the data (in thick lines) with a slope equal to $2H=2-\alpha/2$ as given by Eq. (20), and at large times t the data approach the straight lines. The data curves are scaled by suitable factors for visual clarity.

velocity correlation in Fourier k space with a k dependent prefactor as follows:

$$\overline{\tilde{v}(k_1)\tilde{v}(k_2)} = c(ka)|k_1|^{\alpha-1}\delta(k_1+k_2), \tag{24}$$

with constant $c(ka)=[\cos(ka)\int_{ka}^{\infty}dp(\cos p)/p^{\alpha} + \sin(ka)\int_{ka}^{\infty}dp(\sin p)/p^{\alpha}]$, in the real y space we have $\overline{v(y_1)v(y_2)}$ given by Eq. (5) up to constant factor. The factor $c(ka)|k_1|^{\alpha-1}$ in Eq. (24) is the square of the variance of the distribution of $\tilde{v}$ in k space. In our numerics, we first generate a realization of Gaussian distributed random numbers $\tilde{v}(k)$ in k space with zero mean and variance proportional to $\sqrt{c(ka)}|k|^{(\alpha-1)/2}$, and then Fourier transform back to real space. The latter realization of random $v(y)$ is thus guaranteed to have the required power law correlation in real y space.

We note that to see the correct asymptotic behavior, one has to explore distance scales $y \ll a$. We consider a grid along the y direction with grid spacing Δy, extending from $-N=-y_{max}/\Delta y=-500\,000$ to $N=y_{max}/\Delta y=500\,000$. At each point of this grid we choose $v(y)$ by a discrete Fourier transform of the random numbers $\tilde{v}(k_m)$ obtained for all k_m (with $k_m=-N,\ldots,N$). Once a set of $\{v(y)\}$ is thus chosen, they remain fixed at all times during different thermal histories. This set $\{v(y)\}$ constitutes a particular realization of disorder. Finally one performs the disorder average $\overline{(\cdots)}$ by averaging over various realizations of the set $\{v(y)\}$. In all our simulations we have averaged over 100 realizations of disorder using 1000 thermal realizations for each set of disorder.

There are three further important things to be noted. First, for $\alpha < d$, the role of the cutoff a is unimportant and hence it may be safely ignored. We have verified for $d=1$ that the results are unchanged in comparison to finite a, by taking $a \to 0$, which is equivalent to setting $c(ka)=1$ in Eq. (24). However, the role of a in the $\alpha > d$ regime cannot be overemphasized. The cutoff a has to be chosen such that $a > \Delta y$. We have used $a=10\Delta y$ in our simulations. For low values of the ratio $a/\Delta y$, the expected behavior of $C(t) \sim t^{2-d/2}$ is not seen. The second point is that the value $k=0$ leads to numerical difficulties for $\alpha < d$. The latter was taken care of by using a very small finite value of k in place of $k=0$. Finally we note that the two integrals in the constant $c(ka)$ are rapidly decaying integrals and better results are obtained by using very small grids of integration.

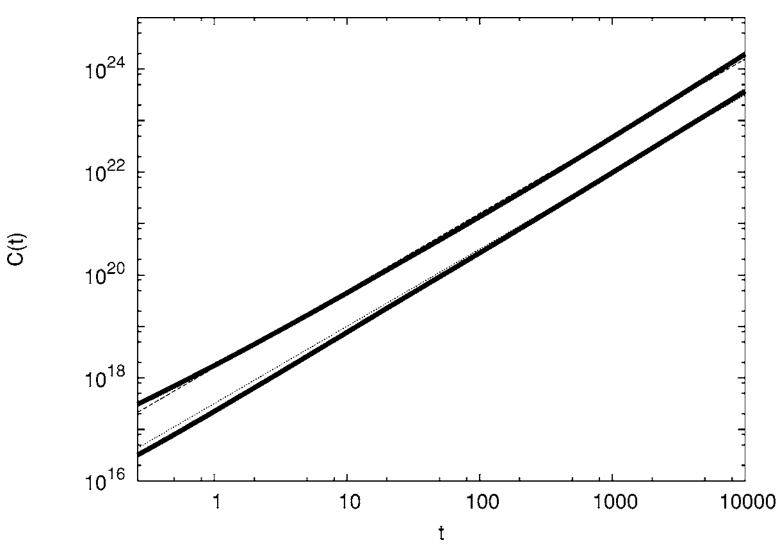

FIG. 3. Log-log plot of $C(t)$ versus t for values of α=3.5 (above) and 1.5 (below). The thin dotted straight lines are fits to the data (in thick lines) with a common slope equal to $2H=3/2$, as in Eq. (20). Both the curves approach this slope at large times. The curves are scaled for visual clarity.

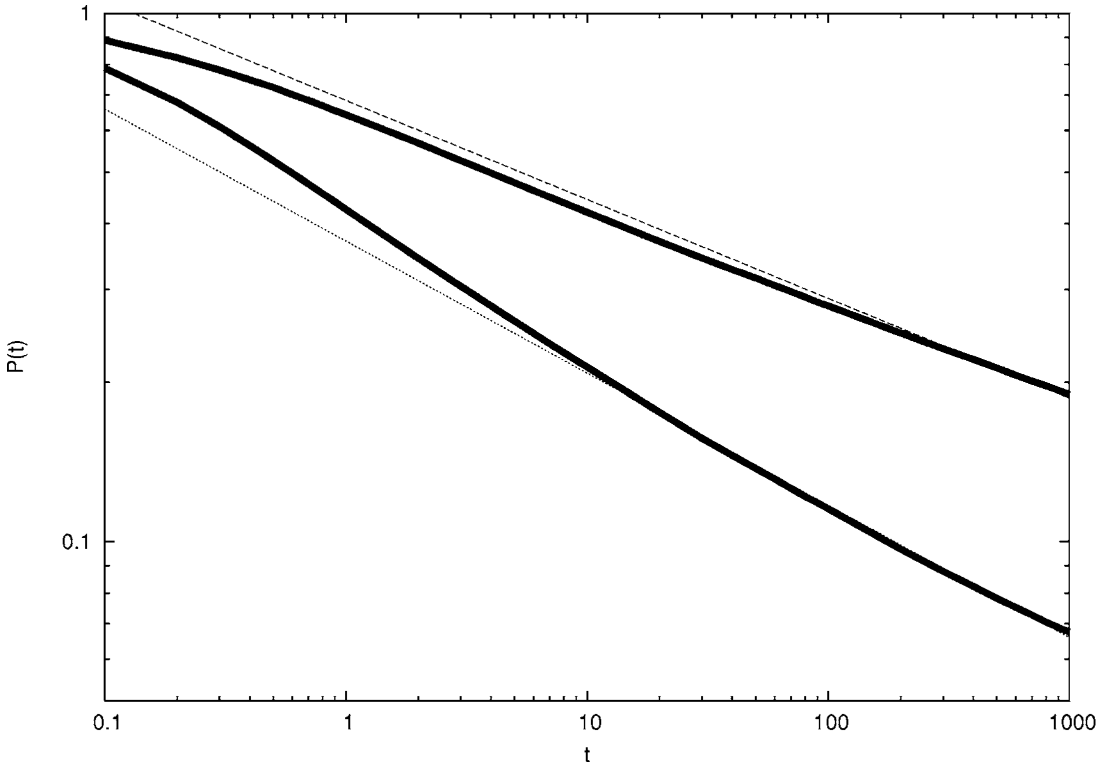

FIG. 4. Log-log plot of $P(t)$ versus t for the values of $\alpha=0.75$ (above) and 3.5 (below). The straight lines are fits to the data with a slope equal to $-\theta$ where $\theta=1-H$ [see Eq. (21)].

In Fig. 2, we see that for different $\alpha<d$ (=1), the incremental correlation function $C(t)\sim t^{2H}$ with exponent $H=1-\alpha/4$, as predicted by Eq. (20). In Fig. 3, we see that for different $\alpha>d$ (=1), again $C(t)\sim t^{2H}$, but now with $2H=2-d/2=3/2$.

We directly measured the averaged persistence probability $P(t)$ in our simulations. The procedure is described clearly in [16]. Our results are shown in Fig. 4. We have shown $P(t)$ for two different α values: for $\alpha<d$ (=1), the power law exponent $\theta=\alpha/4$ as expected, while for $\alpha>d$ (=1) we have $\theta=d/4=1/4$.

V. REMARKS AND CONCLUSION

In this paper, we have derived the incremental two-time correlation function $C(t)$ and persistence probability $P(t)$ of the longitudinal x coordinate of a random walker subjected to a quenched random power law correlated velocity field. This problem is a variant of the original MdM model, now with long-range velocity correlation. The main result is shown in Fig. 1. There are three distinct regions in the parameter space of d versus α. (i) For low dimensions ($d<2$) and large power law exponents ($\alpha>d$), the behavior of $C(t)$ and $P(t)$ are the same as in the short-range correlated MdM model [11]. Thus this regime is *short-range disorder* dominated. (ii) On the other hand for strong long-range disorder, i.e., $\alpha<\min(d,2)$, the temporal behavior of $C(t)$ and $P(t)$ depend on α continuously, so that long-range correlation really has a different effect. The latter regime is *long-range disorder* dominated. (iii) Finally for $d\geqslant 2$ and $\alpha\geqslant 2$, the Hurst exponent is $1/2$, i.e., the same as that of a simple random walker (SRW). Quenched disorder is irrelevant, and this regime is thus *thermal noise* dominated. Further, at the boundaries in Fig. 1 denoted by the dotted lines, there are logarithmic corrections to the power law expressions for $C(t)$ and $P(t)$.

We note that the quenched random velocity field is having an effect on the temporal behavior of a SRW only when the disorder is sufficiently strong and the transverse dimension d is small. For the short-range correlated MdM model [11] H changes only when the number of transverse directions $d<2$. It is as if the velocity kicks in the x direction fail to have much impact when the particle has many more transverse directions to explore. In this paper we find that for $\alpha<\min(d,2)$, i.e., for more strongly correlated or longer-ranged disorder, the short-range correlated MdM model exponents give way to new exponents. Thus even if transverse dimensions are large, sufficiently long-range disorder makes the random walker move superdiffusively. For high dimension ($d>2$) and weak disorder ($\alpha>2$), the quenched noise loses out to the thermal noise, and the asymptotic temporal properties are that of simple Brownian motion.

We also note that the short-distance cutoff scale a of the power law plays a crucial role in this problem. For the case $\alpha>d$, numerically, one has to be very careful to explore distance scales much smaller compared to the power law cutoff scale a. It is one of the most important aspects of this regime that very short-distance correlations seem to control the long-time behavior. For the regime of $\alpha<d$, since correlations at scale $y'>a$ determine the asymptotic behavior rather than $y'<a$, the cutoff a may be ignored.

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